Total No. of Printed Pages—7

# 4 SEM TDC MTMH (CBCS) C 10

**2024**

( May/June )

MATHEMATICS

( Core )

Paper : C–10

**( Ring Theory and Linear Algebra—I )**

*Full Marks* : 80

*Pass Marks* : 32

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

1. (a) Define an integral domain with an example. 1+1=2

(b) Let $R$ be a ring with unity 1. Show that

$$(-1)a = -a = a(-1), \ \forall a \in R$$ 2

(c) Prove that a field has no proper ideals. 3

(d) Prove that in a finite commutative ring with unity, every prime ideal is maximal. 3

(e) Answer any *two* of the following questions : $5 \times 2 = 10$

(i) Let $R$ be the ring of $2 \times 2$ matrices having the elements as real numbers. Then show that the set of matrices of the type

$$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$$

with $a$ and $b$ as real numbers is a subring of $R$. Give an example of a subring which is not an ideal.

(ii) Let $R$ be a commutative ring with unity and $S$ be an ideal of $R$. Show that $\frac{R}{S}$ is an integral domain if and only if $S$ is prime.

(iii) Show that each pair of elements in a principal ideal domain has the greatest common divisor.





2. (a) Define kernel of a homomorphism. 1

(b) Let $C$ be the ring of complex numbers. Is the map $f : C \to C$ such that

$$f(x+iy) = x - iy$$

where $x$ and $y$ are reals, a ring homomorphism? Justify. 2

(c) Let $R$ and $R'$ be two rings and $f : R \to R'$ be a ring homomorphism. Show that—

(i) $f(0) = 0'$, where 0 is zero element of $R$ and $0'$ is zero element of $R'$;

(ii) $f(-a) = -f(a),\ \forall a \in R$. 2+2=4

(d) Determine all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ where $\mathbb{Z}$ is the ring of integers. 3

(e) State and prove the first theorem of isomorphism. 5

*Or*

If $S$ is an ideal of a ring $R$ and $T$ is any subring of $R$, then show that

$$\frac{S+T}{S} \cong \frac{T}{S \cap T}$$

3. (a) Define a vector space. 2

(b) Prove that a subset of a linearly independent set is linearly independent. 2

(c) Does the set $\{(1, 1, 1), (1, 2, 3), (2, -1, 1)\}$ form a basis for $\mathbb{R}^3$? Justify. 3

(d) Let $W$ be a subspace of $\mathbb{R}^4$ spanned by

$$\{(1, -2, 5, -3), (2, 3, 1, -4), (3, 8, -3, -5)\}$$

Find a basis and dimension of $W$. 3

(e) Let $W_1$ and $W_2$ be two subspaces of $V$. Then show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$ 5





*Or*

Let $V$ be a vector space of $n$-dimension and $W$ be a subspace of $V$. Show that any basis $\{W_1, W_2, \cdots, W_k\}$ of $W$ can be extended to a basis $\{V_1, V_2, \cdots, V_n\}$ of $V$ such that $V_i = W_i$, $\forall\, 1 \le i \le k$.

4. (a) What is the range space of a linear transformation? 1

(b) Prove that the map $T : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$T(x, y, z) = (x, y), \quad \forall (x, y, z) \in \mathbb{R}^3$$

is a linear map. 2

(c) Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x+y, y+z, z+x), \forall (x, y, z) \in \mathbb{R}^3$$

Show that $T$ is one-one and onto. 2

(d) Find $\operatorname{Im} T$ and $\ker T$, where $T$ is a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x+2y-z, y+z, x+y-2z), \forall (x, y, z) \in \mathbb{R}^3$$

3+2=5

**5.** Answer any *four* of the following questions : 5×4=20

(a) Let $V$ and $W$ be two finite dimensional vector space, over a field $F$. Show that $V$ and $W$ are isomorphic if and only if

$$\dim(V) = \dim(W)$$

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by

$$T(x, y, z) = (x+z, x-z, y), (x, y, z) \in \mathbb{R}^3$$

Prove that $T$ is invertible and find $T^{-1}$.

(c) Let $V$ and $W$ be two vector spaces over a field $F$, and let $T : V \to W$ be linear. Show that $T^{-1} : W \to V$ is linear if $T$ is invertible.

(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be linear and

$$T(x, y, z) = (4x, 3y, -2z)$$

and $$S(x, y) = (-2x, y)$$

Find $ST$.



( Continued )



(e) Let

$$\beta_1 = \{(1, 0), (0, 1)\}$$

and $$\beta_2 = \{(1, 2), (2, 3)\}$$

be two bases of $\mathbb{R}^2$. Find the transition matrix $P$ from basis $\beta_2$ to basis $\beta_1$.

(f) Let $V$ and $W$ be two vector spaces and $T : V \to W$ be a linear map. Show that

$$\dim V = \text{rank } T + \text{nullity } T$$

(g) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be such that

$$\phi(x, y) = (x-y, x+y), (x, y) \in \mathbb{R}^2$$

and $\phi$ be linear. Prove that $\phi$ is an isomorphism.

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