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4 SEM TDC GEMT (CBCS) 4.1/4.2/4.3

2023

(May/June)

MATHEMATICS

(Generic Elective)

Paper : GE–4.1/4.2/4.3

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

Paper : GE–4.1

(Algebra)

UNIT—1

1. Let n be a positive integer. If n is even, is an n-cycle an odd or an even permutation? Write your answer. 1

2. Justify that in a group, the left and right cancelation laws hold. 2

3. Is the set $\{0, 1, 2, 3, 4, 5, 6\}$ a group with respect to multiplication modulo 7? Give reason. 1+2=3

4. Let G be a group with the following property :

   If $a, b, c \in G$ and $ab = ca$, then $b = c$

   Prove that G is Abelian. 3

5. Mention the symmetries of a rectangle. 4

6. Show that the set $\{5, 15, 25, 35\}$ is a group with respect to multiplication modulo 40. Find the identity element and inverses of each element. 5

7. Compute product of cycles (147)(78)(257) that are permutations of

   $\{1, 2, 3, 4, 5, 6, 7, 8\}$ 5

   Or

   Explain the quaternion group.

8. Let

$$G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} : 0 \neq a \in R \right\}$$

show that G is a group under matrix multiplication. 5

Or

Show that the set of all the cube roots of unity forms a group under usual multiplication of complex numbers.

UNIT—2

9. Prove that in any group an element and its inverse have the same order. 2

10. Let G be a group and $x \in G$. If $x^2 \neq e$ and $x^6 = e$, prove that $x^4 \neq e$ and $x^5 \neq e$; e being the identity of G. 1+1=2

11. Let G be an Abelian group with odd number of elements. Show that product of all the elements of G is the identity. 3

12. Show that a group that has only a finite number of subgroups must be a finite group. 4

13. Show that if a finite group G contains a proper sub-group of index 2 in G, then G is not simple. 5

Or

Let $G = GL(2, R)$ and

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a \text{ and } b \text{ are non-zero integers} \right\}.$$

Prove or disprove H is a sub-group of G.

14. Define normal sub-group. Show that if a finite group G has exactly one sub-group H of a given order, then H is a normal sub-group of G. 1+5=6

Or

Define commutator sub-group. Prove that commutator sub-group of a group G is normal in G. 1+5=6

15. Define a normal sub-group. Show that if H and N are sub-groups of a group G and N is normal in G, then $H \cap K$ is normal in H. 6

Or

Define centre of a group. Let Z be the centre of a group. Prove that if $\frac{G}{Z}$ is a cyclic group, then G is Abelian. 5+1=6

UNIT—3

16. State *True* or *False* : 1

Every ring has an additive identity.

17. State *True* or *False* : 1

A divisor of zero in a commutative ring with unity can have no multiplicative inverse.

18. Show that a unit of a ring divides every element of the ring. 2

19. Let R be a ring and a be a fixed element of R. Show that $I_a = \{ x \in R : ax = 0 \}$ is a subring of R. 3

20. Suppose that R is ring and that $a^2 = a$ for all a in R. Show that R is commutative. 3

21. Show that intersection of subrings of a ring is again a subring of R. 4

22. Show that the set of all purely imaginary numbers under usual addition and multiplication forms a ring. 5

Or

Suppose b and c belong to a commutative ring and bc is a zero divisor. Show that either b or c is a zero divisor.

23. Show that the quaternions form a skew field. 5

Or

Show that the sum of all the elements of a finite field is zero.

(*Continued*)

Paper : GE–4.2

(Application of Algebra)

1. *(a)* দেখুওৱা যে, এটা (v, k, λ)-BIBD, প্ৰতিটো বিন্দু সঠিকভাৱে $r = \dfrac{\lambda(v-1)}{k-1}$ ব্লকত উপলব্ধ, য'ত r অক BIBD ৰ ৰিপ্লিকেশ্যন সংখ্যা বুলি জনা যায়। 5

Show that in a (v, k, λ)-BIBD every point occurs in exactly $r = \dfrac{\lambda(v-1)}{k-1}$ blocks where r is called the replication number of BIBD.

(b) দেখুওৱা যে এটা (v, k, λ)-BIBD ত সম্পূৰ্ণ সঠিকভাৱে $b = \dfrac{vr}{k} = \dfrac{\lambda(v^2 - v)}{k^2 - k}$ সংখ্যক ব্লক থাকে। 5

Show that a (v, k, λ)-BIBD has exactly $b = \dfrac{vr}{k} = \dfrac{\lambda(v^2 - v)}{k^2 - k}$ blocks.

(c) ইন্সিডেন্স কোটিৰ সংজ্ঞা দিয়া। ধৰাহওক $M = (m_{ij})$; $N = (n_{ij})$ দুয়োটা $v \times b$ ইন্সিডেন্স মেট্ৰিক্স। দেখুওৱা যে, দুয়োটা ডিজাইন আইচ'মৰফিক হ'ব যদি আৰু যদিহে, γ আৰু β যথাক্ৰমে $\{1, 2, \ldots, v\}$ আৰু $\{1, 2, \ldots, b\}$ ৰ একোটাহঁত বিন্যাস হয় যাতে, $m_{ij} = n_{\gamma(i), \beta(j)}$; $1 \le i \le v$; $1 \le j \le b$. 6

(*Turn Over*)

Define incidence matrix. Let $M=(m_{ij})$ and $N=(n_{ij})$ be both $v \times b$ incidence matrices. Show that the two designs are isomorphic if and only if there exists permutations γ of $\{1, 2, ..., v\}$ and β of $\{1, 2, ..., b\}$ such that $m_{ij} = n_{\gamma(i), \beta(j)}$; $1 \le i \le v$; $1 \le j \le b$.

2. (a) ধৰা হওক H এটা পেৰিটি ছেক্ মেট্ৰিক্স যাৰ বৈখিক ক'ড হ'ল $C \neq \{0\}$. দেখুওৱা যে Cৰ আটাইতকৈ কম দূৰত্ব হ'ব d, যি সকলোতকৈ ডাঙৰ অখণ্ড সংখ্যা যাতে Hৰ প্ৰতিটো $d-1$ স্তম্ভকৰ সংহতি বৈখিকভাৱে স্বতন্ত্ৰ। 5

Let H be a parity-check matrix of a linear code $C \neq \{0\}$. Show that the minimum distance of C is the largest integer d such that every set of $d-1$ columns in H is linearly independent.

(b) দেখুওৱা যে $F = GF(q)$ত বৈখিক ক'ড $[n, k, d]$ৰ পৃথক জেনেৰেটৰ মেট্ৰিক্সৰ সংখ্যা হ'ব $\Pi_{i=0}^{k-1}(q^k - q^i)$. 5

Show that the number of distinct generator matrix of a linear $[n, k, d]$ code over the Galois field of size q, $F = GF(q)$ is $\Pi_{i=0}^{k-1}(q^k - q^i)$.

(c) চাইক্লিক ক'ডৰ বৈখিক ধৰ্ম আৰু চাইক্লিক স্থানান্তৰকৰণ ধৰ্মৰ বিষয়ে ব্যাখ্যা কৰা। 6

Describe the linear property and property of cyclic shifting of cyclic codes.

3. (a) চমু টোকা লিখা (যি কোনো এটা) : 5

Write short notes on (any *one*) :

(i) প্ৰতিফলন সমমিতা

Reflection symmetry

(ii) ঘূৰ্ণীয়মান সমমিতা

Rotational symmetry

(b) ডাইহিড্ৰেল গ্ৰুপৰ ওপৰত এটা বৰ্ণনাত্মক টোকা লিখা। 5

Write a descriptive note on Dihedral Group.

(c) এটা সংহতিৰ ওপৰত গ্ৰুপ একশ্বনৰ সংজ্ঞা দিয়া। যদি X এটা লেফ্ট G-সংহতি হয় তেন্তে দেখুওৱা যে, যি কোনো $g \in G$ৰ বাবে, ফলন $X \to X$টো যাৰ সংজ্ঞা $x \to g \cdot x$, Xৰ এটা বিন্যাস। 6

Define group action on a set. If X be a G left set, then show that for any $g \in G$, the function $X \to X$ defined by $x \to g \cdot x$ is a permutation of X.

4. (a) আইডেমপটেন্ট, নীলপটেন্ট আৰু ইনভল্যুটৰি কোটিৰ সংজ্ঞা আৰু উদাহৰণ দিয়া। 6

Define idempotent, nilpotent and involutory matrices, with examples.

(b) দেখুওৱা যে তলৰ আকাৰটো পজিটিভ ডেফিনিট : 5

Show that the following quadratic form is positive definite :

$$x^2 + 2y^2 + 3z^2 + 2xy + 4yz + 2zx$$

(c) তলৰ দ্বিঘাতীয় আকাৰটো নৰ্মেল আকাৰলৈ নিয়া : 5

Reduce the following quadratic form into normal form :

$$x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$$

5. (a) তলৰ মেট্ৰিক্সটো বিদিউসদ এশ্বিলন ফৰ্মলৈ নিবলৈ ৰ'-বিদাকশ্যন এলগ'ৰিথম ব্যৱহাৰ কৰা। 8

Use row-reduction algorithm to reduce the following matrix into row-reduced echelon form.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

(b) মেট্ৰিক্স A-ৰ LU-ফেক্টৰাইজেশ্যন নিৰ্ণয় কৰা। 8

Find LU-factorization of the following matrix A.

$$A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix}$$

Paper : GE–4.3

(**Combinatorial Mathematics**)

1. Answer the following questions : 1×4=4

(a) State the principle of inclusion and exclusion.

(b) Write *True* or *False* :

If ${}^nC_{k_1} = {}^nC_{k_2}$, then always $k_1 = k_2$.

(c) Define generating function for a sequence.

(d) How many initial conditions are required to solve the following recurrence relation?

$$a_n = 3a_{n-1} + a_{n-2} - 4a_{n-3}$$

2. Answer the following questions : 2×12=24

(a) Find the number of ordered pairs (x, y) of integers such that $x^2 + y^2 \le 5$.

(b) In how many ways can 5 boys and 3 girls be seated in a table if no girls are adjacent?

(c) Show that ${}^nP_r = (n+1-r)\,{}^nP_{r-1}$.

(d) Let $r \in \mathbb{N}$ s.t.

$$\frac{1}{\binom{9}{r}} - \frac{1}{\binom{10}{r}} = \frac{11}{6\binom{11}{r}}$$

Find the value of r.

(e) Show that $\sum_{r=1}^{n} r\binom{n}{r} = n \cdot 2^{n-1}$.

(f) Among any group of 3000 people, show that there are at least 9 who have same birthday.

(g) Let A, B and C be finite sets. Show that $n(\bar{A} \cap B) = n(B) - n(A \cap B)$.

(h) Each of the 3 boys tosses a die once. Find the number of ways for them to get a total of 14.

(i) Show that the exponential generating function for the sequence (1, 1.3, 1.3.5, 1.3.5.7, ...) is $(1-2x)^{\frac{-3}{2}}$.

(j) Solve : $a_n = 2a_{n-1}$, given that $a_0 = \frac{1}{2}$.

(k) Let $S(n, k)$ be the stirling number of second kind. Prove that $S(n, 2) = 2^{n-1} - 1$ and

$$S(n, n-1) = \binom{n}{2}$$

(l) Determine the cycle index of the alternative group $A(n)$.

3. Answer any *seven* of the following questions : 4×7=28

(a) Find the generating function for the sequence

$$\left\{\binom{n-1}{0}, \binom{1+n-1}{1}, \ldots\ldots, \binom{r+n-1}{r}, \ldots\right\}$$

(b) Prove that

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

where $k, n \in \mathbb{N}$ and $n \geq k$.

(c) Show that

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}$$

for all $m, n, r \in \mathbb{N}$.

(d) Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is at most $\sqrt{2}$.

(e) Find the number of non-negative integer solutions to the equation $x_1 + x_2 + x_3 = 15$, where $x_1 \le 5$, $x_2 \le 6$ and $x_3 \le 7$.

(f) In how many ways can 4 of the letters from PAPAYA be arranged?

(g) Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2})$$

given that $a_0 = 1$ and $a_1 = 0$.

(h) If in a BIBD, $D(v, b, r, k, \lambda)$, b is divisible by r, then show that $b \ge v + r - 1$. (Hint : as b is divisible by r, so $b = nr$ for some n, and for a BIBD $\lambda(v-1) = r(k-1)$).

4. Answer any *four* of the following questions : 6×4=24

(a) What do you mean by symmetric BIBD? Prove that, in case of a symmetric BIBD, any two blocks have λ treatment in common. 1+5=6

(b) Solve the recurrence relation $a_n - 7a_{n-1} + 15a_{n-2} - 9a_{n-3} = 0$, given that $a_0 = 1$, $a_1 = 2$ and $a_2 = 3$. 6

(c) Show that the number of non-negative integer solution of $x_1 + x_2 + \cdots + x_n = r$ is given by

$$\binom{r+n-1}{r}$$ 6

(d) Express the generating function for each of the following sequences in closed form : 3+3=6

(i) $c_r = 3r + 5$ for each $r \in \mathbb{N} \cup \{0\}$

(ii) $c_r = r^2$ for each $r \in \mathbb{N} \cup \{0\}$

(e) How many different necklaces having five beads can be formed using three different kinds of beads, if—

(i) both flips and rotations;

(ii) rotations only? 3+3=6

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