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4 SEM TDC MTMH (CBCS) C 10

2023

(May/June)

MATHEMATICS

(Core)

Paper : C-10

(Ring Theory and Linear Algebra—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. (a) Define unit element in a ring. Give an example. 1+1=2
- (b) Let R be a ring and $a, b \in R$. Show that $a(-b) = (-a)b$. 2
- (c) Show that intersection of two subrings of a ring R is a subring of R . 3

(2)

(d) Show that the ring \mathbb{Z}_p of integers modulo p (p being a prime) is a field. 3

(e) Let R be a commutative ring with unity and let A be an ideal of R . Show that $\frac{R}{A}$ is a field if and only if A is maximal. 5

Or

Define a principal ideal domain. Show that the ring \mathbb{Z} of integers is a principal ideal domain. 1+4=5

(f) Define characteristic of a ring. Let R be a ring with unity. Then show that

characteristic of $R =$

$$\begin{cases} n, & \text{if } 1 \text{ has order } n \text{ under addition} \\ 0, & \text{if } 1 \text{ is of infinite order under addition} \end{cases}$$

$$1+2+2=5$$

Or

Prove that the ring

$$\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}, d \text{ is a positive integer}\}$$

is a field. 5

2. (a) Give an example of ring homomorphism. 1

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(Continued)

(3)

(b) Is the map $f : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}$ such that $f(x) = 2x$ a ring homomorphism? Justify. 2

(c) Let $f : R \rightarrow S$ be a homomorphism from ring R to ring S . If $\ker f = \{0\}$, then show that f is one-one. 2

(d) Let $f : R \rightarrow S$ be a ring homomorphism from ring R to ring S . Show that kernel of f is an ideal of R . 2

(e) Determine all ring homomorphisms from \mathbb{Z} to \mathbb{Z} , \mathbb{Z} being the ring of integers. 3

Or

Show that if R is a ring with unity and $\text{char}(R) = n > 0$, then R contains a subring isomorphic to \mathbb{Z}_n .

(f) Show that if $f : R \rightarrow S$ is an onto homomorphism, then

$$S \cong \frac{R}{\ker f} \quad 5$$

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(Turn Over)

Or

State the second theorem of ring isomorphism. Show that

$$\frac{\mathbb{Z}}{\langle 2 \rangle} \cong \frac{5\mathbb{Z}}{10\mathbb{Z}} \quad / \quad 1+4=5$$

3. (a) Define a vector subspace. 1
- (b) Is $Q(\mathbb{R})$ a vector space? 1
- (c) Is the set $\{(1, 2, 5), (2, -1, 0), (7, -1, 5)\}$ linearly independent over \mathbb{R} ? Justify. 2
- (d) Let $S = \{e_1 - e_2, e_1 + e_2\} \in \mathbb{R}^2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Find $L(S)$. 2
- (e) Show that any basis of a vector space is a linearly independent set. 2
- (f) Show the sum of two subspaces of a vector space V is a subspace of V . 3
- (g) Let V be a vector space and W_1 and W_2 be two subspaces of V . Show that
- $$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \quad / \quad 4$$

Or

Let V be a finite dimensional vector space and W be a subspace of V . Show that $\dim V / W = \dim V - \dim W$.

4. (a) Define a linear map. Show that the map $T : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that
- $$T(x, y, z) = x^2 + y^2 + z^2$$
- is not linear. 1+2=3
- (b) Show that if $T : V \rightarrow W$ is a linear map from vector space V to vector space W , then—
- (i) $T(0) = 0$
- (ii) $T(V)$ is a subspace of W 2+2=4
- (c) Extend the vector $(1, 1, 1)$ in $\mathbb{R}^3(\mathbb{R})$ to form a basis of \mathbb{R}^3 . 3
- (d) Answer any four of the following : $5 \times 4 = 20$
- (i) Let V and W be two vector spaces and $T : V \rightarrow W$ be a linear map. Then show that
- $$\dim V = \text{rank } T + \text{nullity } T$$

(6)

- (ii) Let $\{v_1, \dots, v_n\}$ be a basis of the vector space V and w_1, w_2, \dots, w_n be vectors in the vector space W . Show that there exists a unique linear map $T : V \rightarrow W$ such that

$$T(v_i) = w_i, i = 1, 2, \dots, n$$

- (iii) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by

$$T(x, y) = (x, x + y, y)$$

Then find the range of T .

- (iv) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear operator defined by

$$T(x, y, z) = (3x, x - y, 2x + y + z)$$

Show that T is invertible. Also find the inverse map T^{-1} .

- (v) Let the linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be defined by

$$T(x, y) = (x, y, x + y, x - y)$$

Find the matrix of T with respect to the standard bases.

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(Continued)

(7)

- (vi) Let $T : V \rightarrow W$ be an invertible linear map from vector space V to vector space W where

$$\dim V = \dim W = n$$

Show that T is non-singular if and only if T is onto.

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