Total No. of Printed Pages—7

# 6 SEM TDC MTMH (CBCS) C 14

**2023**

( May/June )

MATHEMATICS

( Core )

Paper : C–14

**( Ring Theory and Linear Algebra—II )**

*Full Marks* : 80

*Pass Marks* : 32

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

**1.** Answer any *three* from the following : 5×3=15

*(a)* Prove that a ring $R$ is a commutative ring with unity if and only if the corresponding polynomial ring $R[x]$ is commutative with unity.

*(b)* If $F$ is a field, then prove that the polynomial ring $F[x]$ is not a field.

(c) Write about irreducibility of a polynomial. Test the irreducibility of the following polynomials : 1+2+2=5

(i) $f(x) = 3x^5 + 15x^4 - 20x^3 + 10x + 20$, over $Q$

(ii) $f(x) = 21x^3 - 3x^2 + 2x + 9$, over $Z_2$

(d) Define principal ideal domain and prove that in a principal ideal domain, an element is an irreducible iff it is prime. 1+4=5

2. Answer any *three* of the following : 5×3=15

(a) Define unique factorization domain (UFD) and prove that every field is unique factorization domain. 1+4=5

(b) Prove that the ring of Gaussian integer $Z[i] = \{a + ib \mid a, b \in Z\}$ is Euclidian domain.

(c) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0 \in Z[x]$. If there is a prime such that $p \nmid a_n$, $p \mid a_{n-1}, \dots, p \mid a_0$ and $p^2 \nmid a_0$, then prove that $f(x)$ is irreducible over $Q$.





(d) Prove that every Euclidian domain is a principal ideal domain.

3. Answer any *three* of the following : 6×3=18

(a) Let $V$ be a finite dimensional vector space over the field $F$. If $\alpha$ is any vector in $V$, the function $L_\alpha$ of $V^*$ defined by $L_\alpha(f) = f(\alpha)$, $\forall f \in V^*$, then prove that $L_\alpha$ is a linear functional and the mapping $\alpha \to L_\alpha$ is an isomorphism of $V$ onto $V^{**}$.

(b) Determine the eigenvalues and the corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(c) Show that similar matrices have the same minimal polynomial. Also, find the minimal polynomial for the real matrix

$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

(d) Let $V$ be a finite dimensional vector space over the field $F$ and $W$ be a subspace of $V$. Then prove that

$$\dim W + \dim W^{\circ} = \dim V$$

4. (a) Let $T: R^2 \to R^2$ be a linear operator defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Then find all the $T$-invariant subspace of $R^2(R)$. 2

(b) Let $T$ be a linear operator on $R^3$ which is represented in the standard basis by the matrix

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

Prove that $T$ is diagonalizable. 5



( Continued )



*Or*

If $T$ is a linear operator on a vector space $V$ and $W$ is any subspace of $V$, then prove that $T(W)$ is a subspace of $V$. Also show that $W$ is invariant under $T$ iff $T(W) \subseteq W$.

5. (a) If $V$ is inner product space, then for any vectors $\alpha, \beta \in V$ and any scalar $c$, prove that—

(i) $\|\alpha\| > 0$ for $\alpha \neq 0$

(ii) $\|c\alpha\| = |c| \, \|\alpha\|$

(iii) $|(\alpha|\beta)| \leq \|\alpha\| \, \|\beta\|$ 5

(b) Apply Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, -1)$, $\beta_3 = (0, 3, 4)$ to obtain an orthonormal basis for $V_3(R)$ with the standard inner product. 5

(c) Let $W$ be any subspace of a finite dimensional inner product space $V$ and let $E$ be the orthogonal projection of $V$ on $W$. Prove that $V = W + W^{\perp}$, where $W^{\perp}$ is the null space of $E$. 5



( Turn Over )

*Or*

If $B=\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ is any finite orthonormal set in an inner product space $V$ and if $\beta$ is any vector in $V$, then prove that

$$\sum_{i=1}^{m}|(\beta, \alpha_i)|^2 \leq ||\beta||^2$$

6. (a) Define orthogonal set. If $\alpha$ and $\beta$ are orthogonal unit vectors, then write the distance between them. 1+1=2

(b) Answer any *two* of the following : 4×2=8

(i) Let $T$ be a linear operator on $R^2$, defined by $T(x, y)=(x+2y, x-y)$. Find the adjoint $T^*$, if the inner product is standard one.

(ii) Let $V$ be a finite dimensional inner product space and let $B=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be an ordered orthonormal basis for $V$. Let $T$ be a linear operator on $V$. Let $A=[a_{ij}]_{n\times n}$ be the matrix of $T$ with respect to ordered basis $B$, then prove that $a_{ij}=(T\alpha_j, \alpha_i)$.



( Continued )



(iii) Let $S=\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ be an orthogonal set of non-zero vectors in an inner product space $V$. If a vector $\beta$ in $V$ is in the linear span of $S$, then show that

$$\beta=\sum_{k=1}^{m}\frac{(\beta, \alpha_k)}{||\alpha_k||^2}\alpha_k$$

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