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1 SEM TDC MTMH (CBCS) C 2

2021

(Held in January/February, 2022)

MATHEMATICS

(Core)

Paper : C–2

(Algebra)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Write the complex number $\sqrt{2}(1+i)$ in the polar form. 1

(b) Find the equation whose roots are the nth power of the roots of the equation $x^2 - 2x\cos\theta + 1 = 0$. 2

(c) Let $\text{cis}\theta = \cos\theta + i\sin\theta$. If $x = \text{cis}\,\alpha$, $y = \text{cis}\,\beta$, $z = \text{cis}\,\gamma$ and $x + y + z = xyz$, then show that

$$1 + \cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = 0$$ 3

Or

If α denotes any nth roots of unity, then show that $1+\alpha+\alpha^2+\dots\dots+\alpha^{n-1}=0$.

(d) Using De Moivre's theorem, find the expansions of $\cos n\theta$ and $\sin n\theta$ where $n \in \mathbb{N}$ and hence deduce the expansions of $\cos\alpha$ and $\sin\alpha$ in powers of α. 4

2. *(a)* State whether true or false : 1

Union of two transitive relations is a transitive relation.

(b) Consider the functions $f:\mathbb{N}\to\mathbb{Z}$ defined by $f(n)=-2n$ and $g:\mathbb{N}\to\mathbb{R}$ defined by $g(n)=\frac{1}{n}$. Investigate the existence of $g\circ f$ justifying your assortion. 1

(c) Show that if $a \equiv b(\text{mod } m)$ and $c \equiv d(\text{mod } m)$, then

$$a+c \equiv b+d \ (\text{mod } m)$$ 2

(d) Define an injective mapping. Show that the mapping $f:\mathbb{Z}\to\mathbb{Z}$ defined by $f(x)=2x$ is injective. 2

(e) Let n be a non-zero fixed integer. For any integers a and b, define a relation $a \equiv b(\text{mod } n)$ if and only if n divides $a-b$. Show that this relation is an equivalence relation. 4

Or

Show that intersection of two equivalence relations on a set is again an equivalence relation.

(*f*) State and prove the well ordering property of the set of positive integers. 4

Or

Show by the principle of mathematical induction that

$$1^2 + 2^2 + ... + n^2 = \frac{n}{6}(n+1)(2n+1)$$

(*g*) Let $f: A \to B$; $g: B \to C$; $h: C \to D$ be mappings. Show that

$$h \circ (g \circ f) = (h \circ g) \circ f$$ 3

(*h*) Let $\text{g. c. d.}(a, b) = 1$. Show that

$$\text{g. c. d.}\,(a+b, a^2 - ab + b^2) = 1 \text{ or } 3$$ 4

(*i*) Let a and b be two integers. Suppose either $a \neq 0$ or $b \neq 0$. Show that there exists a greatest common divisor d of a, b such that $d = ax + by$ for some integers x and y which is uniquely determined by a and b. 4

3. (a) State whether true or false : 1

"Finding the parametric description of the solution set of a linear system is the same as solving the system."

(b) State which of the following statement/statements is/are false : 1

(i) The weights $c_1, c_2, \ldots, c_n$ in a linear combination $c_1v_1 + c_2v_2 + \ldots c_nv_n$ of vectors $v_1, v_2, \ldots, v_n$ can not all be zero.

(ii) Another notation of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is $[a, b]$.

(iii) An example of a linear combination of vectors v_1 and v_2 is $\frac{1}{2}v_1$.

(iv) None of the above are true.

(c) Given $A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & 9 \end{bmatrix}$.

Find one non-trivial solution of $Ax = 0$. 2

(d) Show that the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly dependent. 2

(e) Give the geometrical interpretation of span$\{u, v\}$ where

$$u = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$$

Indicate the subspace represented by the span. 2

(f) Define linear independence of vectors. Show that the columns of the matrix

$$A = \begin{bmatrix} 4 & 3 \\ -2 & -3 \\ 6 & 9 \end{bmatrix}$$ are linearly independent.

1+2=3

(g) Show that if an indexed set $S = \{v_1, \ldots, v_n\}$ with $n \geq 2$, is linearly dependent and $v_1 \neq 0$, then some v_j with $j > 1$ is a linear combination of the preceding vectors $v_1, \ldots, v_{j-1}$. 4

(h) Transform the augmented matrix represented by the linear system,

$$x_1 + 3x_2 + x_3 = 0$$
$$-4x_1 - 9x_2 + 2x_3 = 0$$
$$-3x_2 - 6x_3 = 0$$

(i) to Echelon form indicating the forward phase of row operations.

(ii) to reduced row Echelon form by indicating the backward phase of row operations.

Hence, indicate the basic variables and the free variables. 2+2+1=5

4. (a) Define a linear transformation. 1

(b) Show that $T(0) = 0$ where $T : V \to W$ is a linear transformation. 1

(c) Investigate whether the following transformation is linear or not :

$T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$T(x_1, x_2) = (x_1 + 4, x_2)$ 2

(d) If A is an $n \times n$ invertible matrix, determine the column space of A and null space of A. 2

(e) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Show that T is one-to-one if and only if the equation $T(x) = 0$ has trivial solution. 3

(f) By reducing the matrix

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}$$

to Echelon form, find the number of pivot columns and the rank. 3

Or

Find the characteristic equation of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

and the eigenvalues.

(g) Given a linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = Ax$ where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Find $T(u)$, $T(v)$ and $T(u+v)$ where $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and interpret the effect of the transformation geometrically. 2+2=4

(h) Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$ 4

Or

If $v_1, \ldots\ldots, v_p$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots\ldots, \lambda_p$ of an $n \times n$ matrix A, then show that the set $\{v_1, \ldots., v_p\}$ is linearly independent.

(i) Let A be an invertible matrix. Show that

(i) $(A^{-1})^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$ 2+3=5

Or

Let $v_1, ..., v_p \in \mathbb{R}^n$. Show that the set of all linear combinations of $v_1, ..., v_p$ is a subspace of $\mathbb{R}^n$.

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