Total No. of Printed Pages—20

5 SEM TDC DSE MTH (CBCS)
2.1/2.2/2.3/2.4/(H)

2 0 2 1

(Held in January/February, 2022)

MATHEMATICS

(Discipline Specific Elective)

(For Honours)

Paper : DSE–2.1/2.2/2.3/2.4

The figures in the margin indicate full marks for the questions

Paper : DSE–2.1

(Mathematical Modeling)

Full Marks : 60
Pass Marks : 24

Time : 3 hours

1. (a) What do you mean by an ordinary point of the equation $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$? 1

(b) Define Bessel's equation of order n. 1

2. (a) Determine whether $x=0$ is an ordinary point or a regular singular point of the differential equation

$$2x^2\frac{d^2y}{dx^2}+7x(x+1)\frac{dy}{dx}-3y=0$$ 3

(b) Find the general power series solution near $x=0$ of the Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+p(p+1)y=0$$

where p is an arbitrary constant. 6

Or

Solve the Bessel's equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$ in series, taking $2n$ as non-integral.

3. (a) If $\mathscr{L}[f(t)]$ and $\mathscr{L}[g(t)]$ are Laplace transformations of $f(t)$ and $g(t)$ respectively, then show that

$$\mathscr{L}[af(t)+bg(t)]=a\mathscr{L}[f(t)]+b\mathscr{L}[g(t)]$$

where a and b are constants. 2

(b) If $\mathscr{L}[F(t)]=f(s)$, then prove that

$$\mathscr{L}[F''(t)]=s^2f(s)-sF(0)-F'(0)$$ 3

(c) Evaluate using convolution theorem (any one) : 4

(i) $\mathscr{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

(ii) $\mathscr{L}^{-1}\left[\dfrac{1}{s^2(s+1)^2}\right]$

(d) Apply Laplace transformation and solve

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=e^{-t}\sin t,\ \ y(0)=0,\ \ \left.\frac{dy}{dt}\right|_{t=0}=1$$ 5

4. (a) Write two principal advantages of Monte Carlo simulation technique. 2

(b) Write the algorithm that gives the sequence of calculations needed for a general computer simulation of Monte Carlo technique for finding the area under a curve. 3

5. (a) Who developed the middle-square method for generating random numbers? How does their method work? 1+3=4

(b) Use middle-square method to generate 10 random numbers using $x_0 = 1009$. 3

(c) Use linear congruence method to generate a sequence of 10 random numbers by the rule

$$x_{n+1} = (ax_n + b)\bmod(c)$$

using $a = 5$, $b = 1$ and $c = 8$. 3

6. Write a short note on any *one* of the following : 5

(a) Harbor system simulation algorithm

(b) Morning rush hour queuing model

7. Answer any *one* of the following : 6

(a) A furniture manufacturer makes two products—chairs and tables. Processing of these products is done on two machines P and Q. A chair requires 2 hours on machine P and 6 hours on machine Q. A table requires 5 hours on machine P and no time on machine Q. There are 16 hours per day available on machine P and 30 hours on machine Q. Profit gained by the manufacturer from a chair and a table is ₹ 2 and ₹ 10 respectively. What should be the daily production of each of the two products? (Use graphical method of linear programming model.)

(b) Using Simplex method, solve the following linear programming model :

$$\text{Maximize } 25x_1 + 30x_2$$

subject to

$$20x_1 + 30x_2 \leq 690$$
$$5x_1 + 4x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

8. A company wants to produce three products—A, B and C. The unit profits on these products are ₹4, ₹6 and ₹2 respectively. These products require two types of resources, manpower and raw material. The LP model formulated for determining the optimal product mix is

Maximize $4x_1 + 6x_2 + 2x_3$

subject to the constraints

(i) Manpower constraint

$$x_1 + x_2 + x_3 \leq 3$$

(ii) Raw material constraint

$$x_1 + 4x_2 + 7x_3 \leq 9$$

and $$x_1, x_2, x_3 \geq 0$$

where x_1, x_2, x_3 are number of units of products A, B, C respectively to be produced.

(a) Find the optimal product mix and the corresponding profit of the company. 5

(b) Find the range of the profit contribution of product C in the objective function such that current optimal product mix remains unchanged. 4

Paper : DSE–2.2

(Mechanics)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

1. *(a)* Define moment of a force about a point. 1

(b) Three forces $\vec{P}$, $\vec{Q}$, $\vec{R}$ act along the sides BC, CA, AB of a ΔABC, taken in order; if their resultant passes through the incentre of ΔABC, then prove that

$$\vec{P}+\vec{Q}+\vec{R}=\vec{0}$$ 3

(c) Prove that any system of coplanar forces acting on a rigid body is equivalent to a single force acting at arbitrarily chosen point together with a single couple. 6

Or

Prove that two coplanar couples of equal moments but of opposite sense acting on a rigid body balance each other.

2. *(a)* Write an example of distributed force system. 1

(b) Define equivalent point load. 2

(c) Prove that the necessary and sufficient conditions for the equilibrium of a rigid body under the action of a system of coplanar forces acting at different points of it are that the sums of the resolved parts of the forces in any two mutually perpendicular directions vanish separately and the sum of the moments of the forces about any point in the plane of the forces vanishes. 7

Or

Draw the free body diagram of the beam supported at A by a fixed support and at B by a roller. Explain the significance of each force on the diagram.

Given $W = 40\frac{\text{lb}}{\text{ft}}$ (lb stands for pound)

$a = 3$ ft, $b = 4$ ft and $\theta = 30°$

3. *(a)* Write the Coulomb's law of friction formula. 1

(b) Write the law of Coulomb's friction. 2

(c) A smooth sphere of weight W, rests between a vertical wall and a prism, one of whose face rests on a horizontal plane, if the coefficient of friction between the horizontal and the prism is μ. Show that the least weight of the prism consistent with equilibrium is $W\left(\frac{\tan\alpha}{\mu} - 1\right)$, where α is the inclination to the horizon of the face in contact with the sphere. 7

Or

What is a simple screw jack? A screw jack has a thread of 10 mm pitch. What effort will be required at the handle 40 mm long to lift a load of 2 kN (kilonewton), if the efficiency at this load is 45%?

4. (a) Define first moment of area. 1

(b) Determine first moment of area of a rectangular section of width b and length h about centre of gravity. 3

(c) Write the relation between second moments and products of area. 2

(d) State the transfer theorem for moment of inertia. 2

(e) State and prove Pappus-Guldinus theorem. 7

Or

The lengths AB and CD of the sides of rectangle $ABCD$ are $2a$ and $2b$. Show that the inclination of AB of one of the principal axis at A is

$$\frac{1}{2}\tan^{-1}\left(\frac{3ab}{2(a^2-b^2)}\right)$$

5. (a) Write whether True or False : 1

Fundamental forces like gravity and electric force are conservative.

(b) Write the definition of conservative force field. 2

(c) Prove that the kinetic energy of a system of particles is equal to the kinetic energy of the whole mass moving with the velocity of the centre of mass together with kinetic energy of the particles in their motion relative to the centre of mass. 4

(d) State and prove the principle of conservation of energy. 6

Or

If the sum of the moments of the impulses about a certain line vanishes, then prove that angular momentum about that line remains the same before and after the application of the impulses.

6. (a) Write whether True or False : 1

The change in kinetic energy of a body is equal to the work done by the acting force.

(b) State the laws of rotational motion of a body. 3

(c) If a particle moving in a plane curve under a conservative system of forces, then prove that the sum of the kinetic energy and potential energy is constant. 3

(d) Prove that for rigid body rotating about a point the kinetic energy

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{h}$$

where $\vec{\omega}$ = angular velocity and $\vec{h}$ = angular momentum. 5

Or

Find the moment of momentum of a rigid body rotation about a fixed point.

7. (a) Write the value of time derivative of a unit vector. 1

(b) How is velocity related to the reference frame? 1

(c) What are the three ways of acceleration? 2

(d) State and prove Chasles' theorem. 6

Or

Find the velocity, speed and acceleration of a particle whose motion in space is given by the position vector

(i) $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + (5\cos^2 t)\hat{k}$

(ii) $\vec{r}(t) = (\sec t)\hat{i} + (\tan t)\hat{j} + \frac{4}{3}t\hat{k}$

Paper : DSE–2.3

(Number Theory)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

1. *(a)* Find $\pi(101)$. 1

(b) Solve the linear Diophantine equation

$$172x + 20y = 1000$$

in positive integers. 4

(c) Answer any *one* of the following : 4

(i) Show that the linear Diophantine equation

$$ax + by = c$$

admits a solution if and only if $d \mid c$.

(ii) Using prime number theorem, show that

$$\lim_{x \to \infty} \frac{n \log n}{P_n} = 1$$

where P_n is the nth prime.

2. *(a)* Prove or disprove : 2

For the integers a, b and $n > 1$

$a^2 \equiv b^2 \pmod{n}$ implies $a \equiv b \pmod{n}$.

(*b*) Solve the simultaneous congruences : 4

$$x \equiv 1 \pmod{3}$$
$$x \equiv 2 \pmod{5}$$
$$x \equiv 3 \pmod{7}$$

3. Answer any *one* of the following : 5

(*a*) State and prove Fermat's little theorem.

(*b*) Show that the quadratic congruence

$$x^2 + 1 \equiv 0 \pmod{p}$$

where p is an odd prime, has a solution if and only if

$$p \equiv 1 \pmod{4}$$

4. (*a*) Find $\sigma(10000)$. 1

(*b*) Show that the function τ is multiplicative. 2

5. (*a*) If $p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$ is the prime factorization of the integer $n > 1$, then show that

$$\sum_{d \mid n} \mu(d) \tau(d) = (-1)^j$$

3

Or

Find the highest power of 5 that divides 1000!.

(b) Show that for each integer $n \geq 1$

$$\sum_{d \mid n} \mu(d) = \begin{cases} 0, & \text{if } n > 1 \\ 1, & \text{if } n = 1 \end{cases}$$ 4

Or

Verify that the quadratic residues of 13 are 1, 3, 4, 9, 10 and 12.

6. Show that

$$[x] + [-x] = \begin{cases} 0, & x \text{ is an integer} \\ -1, & x \text{ is not an integer} \end{cases}$$ 2

7. (a) Define Euler's ϕ function. 1

(b) If $p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$ is the prime factorization of the integer $n > 1$, then show that

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_j}\right)$$ 5

Or

State and prove the Möbius inversion formula.

(c) State Euler's theorem. Use Euler's theorem to the unit digit of 3^{100}. 1+3=4

(d) Show that

$$\phi(n) = n \sum_{d \mid n} \frac{\mu(d)}{d}$$ 2

8. (a) Define reduced set of residues modulo n, where $n \geq 1$ is an integer. 1

(b) Show that the integers

$$-5, -4, -3, -2, -1, 1, 2, 3, 4, 5$$

form a reduced set of modulo 11. 2

9. Using Goldbach conjecture, show that for each even integer n, there are integers a and b with $\phi(a) + \phi(b) = n$. 3

Or

Show that $\phi(n)$ is even for the integer $n > 2$.

10. (a) Define a primitive root of the integer n. 1

(b) Show that the only incongruence solutions of $x^2 \equiv 1 \pmod{p}$ are 1 and $p-1$, where p is an odd prime. 3

(c) Let the integer a has order k modulo n. Then show that $a^i \equiv a^j \pmod{n}$ if and only if $i \equiv j \pmod{n}$. 5

Or

Show that there are exactly $\phi(p-1)$ incongruence primitive roots of p, where p is a prime.

11. (a) Determine all the primitive roots of 3^2. 2

(b) Show that the integer 2^k, $k \geq 3$, has no primitive roots. 4

12. Solve $x^2+7x+10 \equiv 0 \pmod{11}$. 4

13. *(a)* Let p be an odd prime and $\gcd(p, a)=1$. Then show that $(a^2/p)=1$. 2

(b) Show that there are infinitely many primes of the form $4k+1$. 3

14. *(a)* Define encrypting. 1

(b) Show that the radius of the inscribed circle of a Pythagorean triangle is always an integer. 5

Or

State and prove Euler's criterion.

Paper : DSE–2.4

(Biomathematics)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

UNIT—I

1. Answer any *two* of the following questions : 7½×2=15

(a) A population is originally 100 individuals, but because of the combined effects of births and deaths, it triples each hour.

(i) Make a table of population size for $t=0$ to 5, where t is measured in hours.

(ii) Give two equations modeling the population growth by first expressing P_{t+1} in terms of P_t and then expressing ΔP in terms of P_t.

(iii) What can you say about the birth and death rates for this population?

(b) In the early stages of the development of a frog embryo, cell division occurs at a fairly regular rate. Suppose you observe that all cells divide and hence the number of cells double, roughly every half-hour.

(i) Write down an equation modeling this situation. You should specify how much real-world time is required by an increment of 1 in t and what the initial number of cells is.

(ii) Produce a table and graph of the number of cells as a function of t.

(c) Obtain a simple prey-predator model, explaining in detail the assumptions taken. Also find the equilibrium positions.

UNIT—II

2. Answer any *two* of the following questions : 7½×2=15

(a) Consider the SI epidemic model. If the contact rate is 0·001 and the number of susceptible is 2000 initially, determine—

(i) the number of susceptible left after 3 weeks;

(ii) the density of susceptible when the rate of appearance of new cases is a maximum;

(iii) the time (in weeks) at which the rate of appearance of new cases is a maximum;

(iv) the maximum rate of appearance of new cases.

(b) In a SIS model, if the infection is spread only by a constant number of carriers, then show that

$$I(t) = \left(I_0 - \frac{\alpha CN}{\alpha C + \beta} e^{[-(\alpha C + \beta)t]} \right) + \frac{\alpha CN}{\alpha C + \beta}$$

where I and C are the number of infectives and carriers; N is total population; α and β are contact rate and susceptible rate respectively; I_0 is the infectives at $t = 0$.

(c) Let x and y respectively denote the proportion of susceptibles and carriers in a population. Suppose the carriers are identified and removed from the population at a rate β, so that $\frac{dy}{dt} = \beta y$.

Suppose also that the disease spreads at a rate proportional to the product of x and y, thus

$$\frac{dx}{dt} = -\alpha xy$$

(i) Determine the proportions of carriers at any time t, where $y(0) = y_0$.

(ii) Use (i) to find the susceptibles at time t, where $x(0) = x_0$.

(iii) Find the proportion of population that escapes the epidemic.

UNIT—III

3. Answer any *two* of the following questions : $7\frac{1}{2} \times 2 = 15$

(a) Consider the competition model for two species with populations N_1 and N_2

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1}\right)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - b_{21} \frac{N_1}{K_2}\right)$$

where only one species N_1 has limited carrying capacity. Investigate their stability and sketch the phase plane trajectories. [Here K_1, K_2 are carrying capacities; r_1, r_2 are linear birthrates of the populations N_1 and N_2 respectively. b_{12}, b_{21} measure the competitive effect of N_2 on N_1 and N_1 on N_2 respectively.]

$4+3\frac{1}{2}=7\frac{1}{2}$

(b) What is Routh-Hurwitz criteria? Explain with reference to multiple species communities. $2+5\frac{1}{2}=7\frac{1}{2}$

(c) Discuss bifurcation and limit cycle with respect to any biological model.

UNIT—IV

4. Answer any *two* of the following questions : $7\frac{1}{2}\times2=15$

(a) Write short note on any *one* of the following :

(i) One species model with diffusion

(ii) Two species model with diffusion

(b) For a blood vessel of constant radius R, length L and driving force $P = P_1 - P_2$, show that the average velocity of the flow is equal to half of the maximum velocity and the resistance is proportional to $\frac{L}{R^4}$.

(c) Consider arterial blood viscosity

$$\mu = 0 \cdot 027 \text{ poise}$$

If the length of the artery is 2 cm and radius 8×10^{-3} cm and $P = P_1 - P_2 = 4 \times 10^3$ dynes/cm^2, then find *(i)* $q_z(r)$ and the maximum peak velocity of blood and *(ii)* the shear stress at the wall.

(Here q_z denote velocity along *z*-axis, p_1 and p_2 denote pressure at two ends of the artery.)

UNIT—V

5. Answer any *two* of the following questions : 10×2=20

(a) Let D and d, and W and w respectively denote allele for tall and dwarf, and round and wrinkled seeds of peas. Find the outcome of the product DdWw × ddWw using Puneet square or using probability. Also find the probability that the progeny of DdWw × ddWw is dwarf with round seeds. 6+4=10

(b) Explain in detail the Hardy-Weinberg equilibrium, mentioning the assumptions considered for the equilibrium.

(c) Compare and contrast stage structure model with age structure model.

★★★