5 SEM TDC MTMH (CBCS) C 12

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(Held in January/February, 2022)

MATHEMATICS

(Core)

Paper : C–12

(Group Theory—II)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Let $H = \{(1), (12)\}$. Is H abelian? 1

(b) Define commutator subgroup and characteristic subgroup. 2+2=4

(c) Prove that if G is a cyclic group, then Aut$G$ is abelian. 3

(d) Let G be a cyclic group of infinite order. Then prove that $O(\text{Aut}G) = 2$. 3

(e) Prove that a group G is abelian if and only if I_G is the only inner automorphism. 3

(f) Let G be a group, then prove that $f: G \to G$ defined by $f(x) = x^{-1}$, $\forall x \in G$ is automorphism if and only if G is abelian. 4

2. Answer any *two* of the following : 6×2=12

(a) Let $I(G)$ be the set of all inner automorphisms on a group G, then prove that—

(i) $I(G)$ is normal subgroup of Aut$G$;

(ii) $I(G) \cong \dfrac{G}{Z(G)}$.

(b) Let G be a cyclic group generated by a and $O(G) = n > 1$, then prove that a homomorphism $f: G \to G$ is an automorphism if and only if $G = \langle f(a) \rangle$.

(c) Let G be a group and G' be the commutator subgroup of G, then prove that—

(i) G' is normal subgroup of G;

(ii) $\dfrac{G}{G'}$ is abelian;

(iii) if N is any normal subgroup of G, then G/N is abelian if and only if $G' \subseteq N$.

(d) Let G be group and $Z(G)$ be the centre of G, then prove that if $\dfrac{G}{Z(G)}$ is cyclic, then G is abelian.

3. *(a)* Define internal direct product. 2

(b) Let $G_1, G_2, \cdots, G_n$ be a finite collection of groups such that

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n = \{(g_1, g_2, \cdots, g_n) : g_i \in G_i\}$$

then prove that

$$|(g_1, g_2, \cdots, g_n)| = \text{lcm}(|g_1|, |g_2|, \cdots, |g_n|)$$ 3

(c) If s and t are relatively prime, then prove that $U(st) \cong U(s) \oplus U(t)$. 4

(d) Suppose that a group is an internal direct product of its subgroups H and K. Then prove that—

(i) H and K have only the identity element in common;

(ii) G is isomorphic to the external direct product of H by K. 5

Or

Prove that a group G is internal direct product of its subgroups H and K if and only if—

(i) H and K are normal subgroups of G;

(ii) $H \cap K = \{e\}$.

(e) If m divides the order of a finite abelian group G, then prove that G has a subgroup of order m. 6

Or

Let G be a finite abelian group of order $p^n m$, where p is prime and $p \nmid m$, then prove that $G = H \times K$ where $H = \{x \in G : x^{p^n} = e\}$ and $K = \{x \in G : x^m = e\}$.

4. *(a)* Write the class equation for the group G. 1

(b) Define sylow p-subgroup and conjugacy class. 2+2=4

(c) If $|G| = p^2$, where p is prime, then prove that G is abelian. 3

(d) Let G be a finite group and $Z(G)$ be the centre of G. Then prove that

$$O(G) = O(Z(G)) + \sum_{a \notin Z(G)} \frac{O(G)}{O(N(a))}$$ 3

(e) Answer any *two* of the following : 4×2=8

(i) Let G be a group. Then prove that $O(C(a)) = 1$ if and only if $a \in Z(G)$.

(ii) Prove that every abelian group of order 6 is cyclic.

(iii) Prove that a group of order 12 has a normal sylow p-subgroup or sylow 3-subgroup.

(Continued)

(f) Prove that no group of order 30 is simple. 5

Or

Prove that a sylow p-subgroup of a group G is normal if and only if it is the only sylow p-subgroup of G.

(g) Suppose that G is a finite group and $p|O(G)$ where p is a prime number, then prove that there is an element a in G such that $O(a) = p$. 6

Or

Let G be a group of finite order and p be a prime number. If $p^m|O(G)$ and $p^{m+1}\nmid O(G)$, then prove that G has subgroup of order p^m.

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